MATH 135 — QUIZ 10 SOLUTIONS — JAMES HOLLAND
2019-11-12

Question 1. Suppose the cost of making $x > 0$ loaves of bread is (in dollars)

$$C(x) = x^3 - 9x^2 + 16x + 1.$$

After making x loaves of bread, suppose each loaf will be sold for $p(x) = 1 + \frac{100}{x}$ dollars. Determine the level of production of loaves that maximizes profit.

Solution .:.

The revenue from selling all x loaves is $x \cdot p(x) = x + 100$. The profit is thus given by $P(x) = x \cdot p(x) - C(x)$, which means we are trying to maximize

$$P(x) = x + 100 - x^3 + 9x^2 - 16x + 99 = -x^3 + 9x^2 - 15x + 99.$$

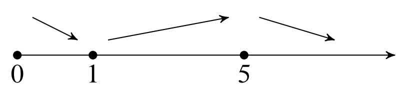
To find the maximum of this, we just need to find the critical points, and then classify them using derivative tests. To find the critical points, note that

$$P'(x) = -3x^2 + 18x - 15 = -3(x^2 - 6x + 5) = -3(x - 5)(x - 1).$$

This is 0 iff $x = 5$ or $x = 1$. These yield critical points of $(5, P(5))$ and $(1, P(1))$. To classify the critical points, we use the first derivative test;

- For $x < 1$, one can see that $(x - 5) < (x - 1) < 0$ and so $P'(x)$ is negative on $(0, 1)$.
- For $1 < x < 5$, one can see that $(x - 5) < 0$ while $0 < (x - 1)$. Hence $P'(x)$ is positive on $(1, 5)$.
- For $x > 5$, $(x - 1) > (x - 5) > 0$ so that $P'(x)$ is negative on $(5, \infty)$.

So we get the following number line that allows us to see that $(1, P(1))$ is a relative minimum while $(5, P(5))$ is a relative maximum.



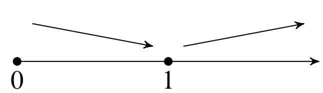
To confirm that $(5, P(5))$ is an actual maximum for $P(x)$ on $x > 0$, we must check the endpoint of 0.[i] Since $P(0) = 99 < P(5) = 124$, it follows that $x = 5$ loaves of bread gives a maximum profit.

Question 2. Suppose it takes $T(x) = x^3 - 2x\ln(x)$ minutes to help $x > 0$ patients. Find the minimum *average* time spent per patient.

Proof .:.

The average time spent is $A(x) = T(x)/x = x^2 - 2\ln(x)$. To find the minimum of this, we find the critical points, and then classify them using derivative tests. $A'(x) = 2x - 2/x$, which is undefined when $x = 0$. But this doesn't correspond to a critical point, since 0 isn't in the domain of A. So the only critical numbers we care about are those that give $0 = A'(x) = 2x - 2/x$, which means $2x = 2/x$ and thus $x^2 = 1$. Since $x > 0$, this means $x = 1$, and so $(1, A(1)) = (1, 1)$ is the only critical point.

To classify the critical points, we use the first derivative test. For $0 < x < 1$, $x < 1/x$ so that $A'(x) < 0$. For $x > 1$, $x > 1/x$ so that $A'(x) > 0$. Hence we get the following number line that allows us to see that $(1, A(1)) = (1, 1)$ is a minimum for A on $(0, \infty)$.



Thus the minimum average time spent per patient is $A(1) = 1$.

[i] The "endpoint" of ∞ as been dealt with by the first-derivative test: $\lim_{x\to\infty} P(x) < P(5)$, because P always decreases after 5.